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The ability to resolve track-to-track association problems is one of the most important requirements in the development of automated information processors for ocean surveillance systems comprising multiple data sources. The individual tracks carried by a processor can be represented as samples from multivariate normal distributions, and the most basic track association problem—determining whether two tracks relate to the same target—is solved using a test based on the chi-square distribution. When a specified track is to be compared with two other candidates for association, a test of the chi-square distribution against a noncentral chi-square alternative is used; this

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procedure can be adapted for use as a sequential test in continuing surveillance operations. The procedures developed for track association can be implemented within man-machine systems; for many cases they will make relatively few demands for interaction by surveillance analysts.

CONTENTS

TRACK-TO-TRACK ASSOCIATION IN OCEAN SURVEILLANCE

INTRODUCTION

The track-to-track association problem in ocean surveillance is discussed in this report. A family of procedures is described for deciding if two observed tracks are from the same target, and these procedures are discussed within the operational context of a large-scale automated surveillance-data processing system.

The Navy's roles in protecting and maintaining the sea lines of communication and in supporting other areas of national concern require a major effort in the surveillance of ocean surface traffic. This effort is required for monitoring possibly hostile combatant ships to protect the U.S. fleet and also for such activities as guaranteeing the freedom of movement of U.S.-bound oil tankers and locating and warning U.S.-flag merchant ships of possible dangers. The increasing need for effective and timely surveillance information has generated a requirement for diverse data sources and for the capability to process surveillance data more rapidly than can be done by unaided analysts. The Navy is developing systems which accept surveillance data of various types from a variety of sources, process the data at speeds adequate to meet increased report arrival rates, and store and maintain the finished product in a readily accessible form. A useful format for stored information is a *track*, defined as a time-ordered sequence of positions, with all sequence elements relating to the same target. Instances of the track-to-track association problem occur when two unidentified tracks are examined to determine whether they could have been developed on the same target or when two tracks identified as relating to the same target are examined to decide if the identification is, in both cases, correct.

This report is one of a series of studies resulting from continuing Naval Research Laboratory efforts in the development and analysis of surveillance systems and concepts. Recent NRL studies of surveillance problems include Refs. 1 through 5. For the case of two sensors, each of which has observed the position of a vessel, Ref. 5 developed a method for determining whether one target or two targets had actually been detected. The present study extends this situation to the case of target tracks. The test procedures discussed in this report are not unique to the track association problem. Reference 6, for example, notes that similar procedures are applicable in surveillance, quality control, and motion detection. The purpose of this report however is to describe the relevance of each of the test procedures' operating parameters to the track association problem and to indicate how these parameters will affect operational implementation of the procedures.

Manuscript submitted February 26, 1976.

RESULTS

The major analytic results of this investigation involve the resolution of track-to-track association problems through a series of statistical tests. The basic problem of determining whether two observed tracks relate to the same target has been formulated as a test involving the chi-square distribution. An extended problem of determining which of two candidate tracks a given track can best be associated with has been formulated as a test of the chi-square distribution against a noncentral chi-square alternative. Finally the extended problem with the added feature that decisions can be deferred pending the receipt of additional data has been formulated as a sequential test. The test for the basic problem can be readily implemented within an automated surveillance-data processing system and can be carried out automatically without requiring inputs from surveillance analysts. The other tests will require inputs from analysts. The estimates required of the analyst are of operationally significant parameters, although fortunately in many cases the final decision of the test is not sensitive to moderate changes in the parameters.

OPERATIONAL SITUATION

For this study, processing of surveillance information consists of maintaining and updating data files on tracked targets. The procedure is assumed to be carried out by a processing system which includes surveillance analysts, computers, and analyst aids, both manual and computer-driven. Computers perform the bulk of the routine data-processing functions, freeing the analysts for making decisions in difficult cases and for providing judgments which are beyond the capabilities of automated processes.

Track association schemes will be used to associate tracks resulting from a report-to-track association process. These tracks must contain the best available information on observed target movements. Report-to-track association processes operate on sets of reported target positions and sets of established tracks and attempt to match correctly the elements of these sets. The pairing of a reported position with an established track extends the track to a new position, either the position observed in the report or a smoothed position obtained from operations on the reported position and track parameters. When two or more tracks exhibit similar positions over an interval of time, a track association scheme is called into use.

The tracks upon which a track association scheme is to be used will contain the underlying positions and reported times, together with the covariance matrices associated with these positions. If the track comprises independently observed positions, then the covariance matrix will indicate no correlation between position coordinates for different observation times. However, if the track is made up of smoothed positions, then nonzero correlations will occur. This is because track smoothing schemes are based on estimating target positions by applying a computational procedure either to previous smoothed positions or directly to previous observed positions, so that each smoothed position contains information about previous smoothed positions; thus the covariance matrix associated with a set of smoothed positions will contain entries indicating correlation between the position estimates.

In this report the surveillance information is considered to be generated by two independently operating surveillance systems. The first system, called the *primary* system, is defined as possessing superior capabilities in a number of important system characteristics, such as high detection probability, high frequency of observation of a given area, or small position estimate errors. The other system (the *secondary* system) is composed of subsystems, no one of which is as effective as the primary system. The members of the secondary system may have poor detection capabilities, may detect targets only through infrequently exposed characteristics, or may possess large localization errors. They may however possess important characteristics which the primary system does not. For example the secondary system might be able to identify the target by nationality, by class (merchant or combatant), or by name. Although tracking unknown targets with the primary system may provide valuable information on target movements, the addition of information on target identity from a secondary source will increase the information content of both tracks and may provide essential data for analyzing tactics or predicting intentions.

It is assumed that throughout the major part of the surveillance operation the data from the two surveillance systems are processed independently and the report-to-track association and track continuation processes that were described are conducted separately for the two systems. Thus prior to the employment of the track-to-track association process the data base of the surveillance processing system contains tracks generated by the primary system and tracks generated by the secondary system. At certain times however the track families are examined to determine whether any of them refer to the same target and thus can be merged into a single track.

BASIC MODEL

In this section the basic model employed in the investigation is described; this model involves one track from the primary system and one track from the secondary system. The primary system is denoted S_1, and its track is denoted T_1; the secondary system is denoted S_0, and its track is denoted T_0.

Figure 1, which shows T_0 and T_1, is representative of the basic problem formulation. The tracks have been generated by connecting target positions as generated by the two sensor systems. In the basic model each track consists of p positions, and the two systems are assumed to have both made their observations at the same times t_1, t_2, ..., t_p. The ith position of track T_k is denoted by the coordinate pair $(x_i^{(k)}, y_i^{(k)})$, for $i = 1, 2, ..., p$ and $k = 0, 1$. Thus each T_k can be associated with the n-by-one (where $n = 2p$) column vector

$$\begin{pmatrix} x_1^{(k)} \\ y_1^{(k)} \\ x_2^{(k)} \\ y_2^{(k)} \\ \cdots \\ x_p^{(k)} \\ y_p^{(k)} \end{pmatrix}.$$

It is assumed that the n-tuples

$$\left(x_1^{(k)}, y_1^{(k)}, \ldots, x_p^{(k)}, y_p^{(k)}\right)$$

are observations from an n-dimensional normal distribution. For an arbitrary random variable x, let $\bar{x}$ represent the expectation of x. Then each observed track T_k is regarded as a sample from an $n = 2p$ dimensional normal distribution

$$\mathcal{N}\left[\begin{pmatrix} \bar{x}_1^{(k)} \\ \bar{y}_1^{(k)} \\ \cdots \\ \bar{x}_p^{(k)} \\ \bar{y}_p^{(k)} \end{pmatrix}; \sum_k\right],$$

where each Σ_k is a $2p$-by-$2p$ covariance matrix. The elements of the matrices Σ_k are determined by the characteristics of the sensor systems and possibly by the analytic methods used to obtain smoothed coordinates $(x_i^{(k)}, y_i^{(k)})$ from the reported observed positions.

The test statistic to be used, denoted R^2, is defined by

$$R^2 = \left(T_0 - T_1\right)' \left(\sum_0 + \sum_1\right)^{-1} \left(T_0 - T_1\right), \tag{1}$$

where the prime indicates matrix transpose. Given Σ_0 and Σ_1, it is known [7] that there exists a nonsingular matrix P such that $P(\Sigma_0 + \Sigma_1) P' = I$, where I is the n-by-n identity matrix. It will be shown that the statistic

$$W^2 = |P(T_0 - T_1)|^2 \tag{2}$$

has a distribution whose characteristics are useful in this investigation. Since it can be shown that $W^2 = R^2$, in the following we will consider that $R^2 = |P(T_0 - T_1)|^2$ and use the notation r^2 for observed values of R^2 or for values which have been calculated from a specified set of track positions.

NULL HYPOTHESIS H_0

The null hypothesis to be tested is H_0: T_0 and T_1 are from the same target. More specifically, let $\overline{T}_k$ represent the column vector of expected values

$$\overline{T}_k = \begin{pmatrix} \overline{x}_1^{(k)} \\ \overline{y}_1^{(k)} \\ \cdots \\ \overline{x}_p^{(k)} \\ \overline{y}_p^{(k)} \end{pmatrix}, \quad k = 0, 1.$$

Then an equivalent formulation of the null hypothesis is

$$H_0: \overline{T}_0 = \overline{T}_1$$

or

$$H_0: \overline{T}_0 - \overline{T}_1 = \underset{\sim}{0}$$

where $\underset{\sim}{0}$ is the $2p$-by-1 column vector with all components equal to 0.

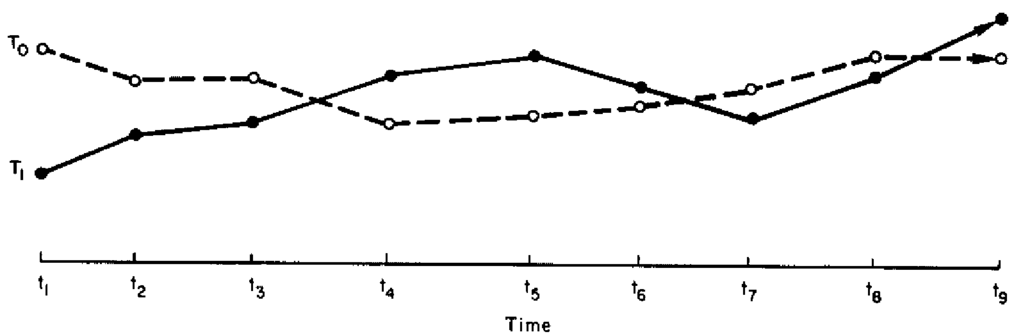

Fig. 1—The track association problem in the basic model, in which the primary system (subscript 1) and secondary system (subscript 0) observe track positions simultaneously

Since the tracks T_0 and T_1 are samples from n-dimensional normal distributions, with $n = 2p$, the difference $T_0 - T_1$ is a sample from an n-dimensional normal distribution with expectation $\overline{T}_0 - \overline{T}_1$ and covariance matrix $\Sigma = \Sigma_0 + \Sigma_1$. As mentioned, under H_0, $T_0 - T_1 = \underset{\sim}{0}$. With the matrix P defined as just before equation (2), it follows by properties of the multivariate normal distribution [8, pp. 24-25] that $P(T_0 - T_1)$ is a sample from an n-dimensional normal distribution with zero mean and identity covariance matrix. Thus [7, p. 56] under H_0, R^2 has the (central) chi-square distribution with $n = 2p$ degrees of freedom, with probability density function (p.d.f.)

$$f_0(r^2; n) = [2^p \Gamma(p)]^{p-1} e^{-r^2/2}. \tag{3}$$

The null hypothesis H_0 is rejected whenever the computed value r^2 of R^2 seems too large. A type-I error occurs when the null hypothesis is incorrectly rejected; the probability of this error, the *level of significance*, is denoted α. For a specified value of α and a given number of observed positions, the acceptance region for H_0 is bounded below by 0 and above by the value χ^2 satisfying the expression

$$1 - \alpha = \int_0^{\chi^2} f_0(r^2; n)\, dr^2 . \tag{4}$$

Values of χ^2 for given α and p are found in tables of the chi-square distribution, such as in Ref. 9; Table 1 contains representative values. For computer application either the chi-square tables can be stored in memory or analytic expressions can be used to compute approximations to χ^2; such approximations are discussed in Chapter 17 of Ref. 10 (Vol. 1).

It is valuable to consider the situation that inspires the rejection of the null hypothesis H_0; these considerations lead to discussions of when the significance test ought to be carried out on two observed tracks. It is clear that the test should be carried out when and only when there is doubt as to the source of two observed tracks. As an example of a situation in which the test is *not* required, suppose there are tracks T_0 and T_1 as shown in Fig. 1, where the primary surveillance system has a high detection probability. For H_0 to be rejected, the track T_0 must be assumed to have come from a target other than that generating T_1. However, as shown in the figure, no such alternate track has been detected, and the existence of an alternate target is a low probability event in the light of the primary system's high detection probability. Thus, since no other target could have generated T_0, it is sufficient to associate the two tracks without recourse to the test.

It is not sensible to carry out the test unless rejection of H_0 is operationally feasible. If the situation at hand does not admit the possibility of at least one additional target for association with T_0, then the two candidate tracks should be associated. Situations in which the test should be carried out will always admit the possibility of separate targets underlying the tracks T_0 and T_1. For example, whenever the primary surveillance system has only a moderate detection probability, it is possible that a target may be present yet not be detected sufficiently often to have generated a track. Another possibility is that a number of candidate tracks have been detected by the primary system, so that should the test fail there will be other candidates for association with T_0. This case is discussed in the following section.

Table 1—Acceptance Regions for the Hypothesis H_0

Number p of Observed Positions in Each Track	Maximum Value χ^2 for Acceptance of H_0				
	$\alpha = 0.30$	$\alpha = 0.20$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$
2	4.878	5.989	7.779	9.488	13.277
4	9.524	11.030	13.362	15.507	20.090
6	14.011	15.812	18.549	21.026	26.217
8	18.418	20.465	23.542	26.296	32.000
10	22.775	25.038	28.412	31.410	37.566
15	33.530	36.250	40.256	43.773	50.892

ALTERNATE HYPOTHESIS H_1

Although the track association test should be performed only when it is reasonable to suggest that track T_0 could be related to a target other than the source of T_1, it is not always the case that a definite alternate source for T_0 can be proposed. It is one thing to accept the possibility of an alternate source, but it is another to specify one. In the situation discussed in this section, an observed track is specified as relating to a possible alternate source for T_0. In this case one can specify an alternate hypothesis H_1 and determine the power of the track association test against this alternative.

Figure 2, adapted from Ref. 11, illustrates an operational situation in which this case could arise. This figure contains examples of ship track histories a surveillance system such as the primary system might obtain over an interval of time for a given area of interest. Each of these tracks represents a separate target. This example contains both ships which appear to be moving randomly and ships which move in an apparently more predictable manner. The illustration shows much track crossing and few cases in which two or more tracks remain parallel for an extended period. Suppose that a track T_0 from the secondary system appears to be related to one of the tracks in this illustration. If the association test should fail, then it would appear that there will be at most one other track with which T_0 might be associated. If the primary system has a relatively high detection probability, then it is unlikely that it would not have detected the target which generated T_0. Consequently it is assumed that there will be another candidate track for T_0 and thus that if association with the first track should fail, then association with the second track should be accepted.

Figure 3 illustrates the extension of the basic model to the case of testing the hypothesis H_0 against a specific alternative. The tracks T_0 and T_1 are as before. Another track T_2 is now present and is assumed to have been obtained from the primary system S_1. The track T_0 must relate to one of the other tracks; if the hypothesis that T_0 is associated with T_1 is rejected, then T_0 will be associated with T_2. The statistical distributions associated with T_0 and T_1 are as was described in the section on the basic model. The observations comprising T_2 constitute a column vector

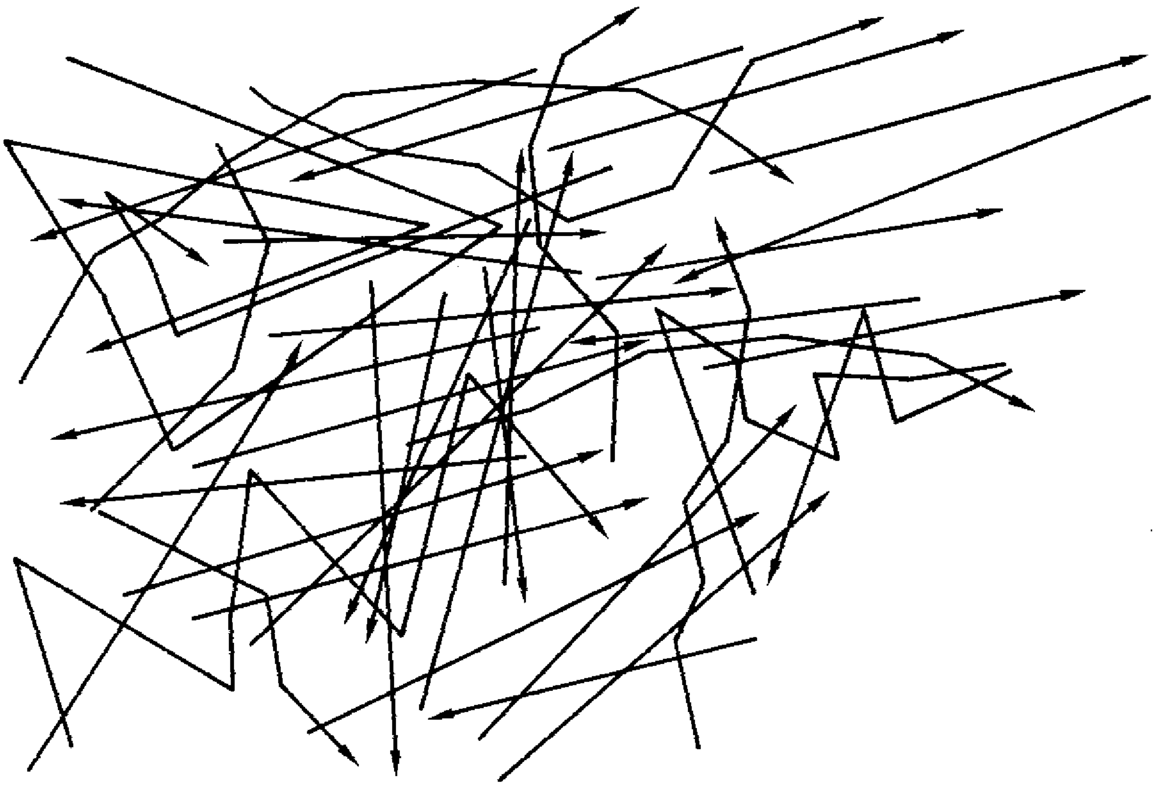

Fig. 2—Ship track histories from a single sensor.

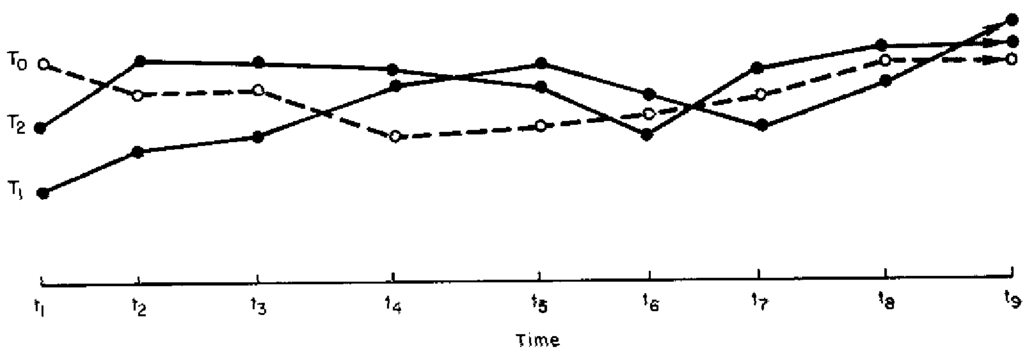

Fig. 3—Track association problem with two candidate tracks.

$$T_2 = \begin{pmatrix} x_1^{(2)} \\ y_1^{(2)} \\ x_2^{(2)} \\ y_2^{(2)} \\ \cdots \\ x_p^{(2)} \\ y_p^{(2)} \end{pmatrix},$$

and this vector is assumed to be a sample from a $2p$-dimensional normal distribution with expectation

$$\overline{T}_2 = \begin{pmatrix} \overline{x}_1^{(2)} \\ \overline{y}_1^{(2)} \\ \overline{x}_2^{(2)} \\ \cdots \\ \overline{x}_p^{(2)} \\ \overline{y}_p^{(2)} \end{pmatrix}$$

and $2p$-by-$2p$ covariance matrix Σ_2. It is clear that this case admits a specified alternate hypothesis H_1, which states that T_0 is related to T_2. The alternate is given by

$$H_1 \colon \overline{T}_0 = \overline{T}_2 .$$

Distribution of R^2 Under H_1; Power of the Test

Under H_1 the difference vector $T_0 - T_1$ is a sample from a $2p$-dimensional normal distribution with expectation $\overline{T}_2 - \overline{T}_1$ and covariance matrix $\Sigma = \Sigma_0 + \Sigma_1$. Thus [7, p. 56] R^2 has the *noncentral chi-square distribution* with $n = 2p$ degrees of freedom and noncentrality parameter

$$\begin{aligned} \lambda &= (\overline{T}_2 - \overline{T}_1)' \; \Sigma^{-1}(\overline{T}_2 - \overline{T}_1) \\ &= |P(\overline{T}_2 - \overline{T}_1)|^2 , \end{aligned}$$

where matrix P is as defined previously. The probability density function of the noncentral chi-square distribution with n degrees of freedom and noncentrality λ is defined by

$$f_1(r^2; n, \lambda) = \frac{1}{2}\left(\frac{r^2}{\lambda}\right)^{(n-2)/4} e^{-(r^2+\lambda)/2} I_{(n-2)/2}(r\sqrt{\lambda}), \quad r \geqslant 0,$$

where $I_K(\ldots)$ is the modified Bessel function of the first kind and order K [Ref. 10]. Thus for a given set of p positions, a specified value for the level of significance α, and specified value of λ, the probability of incorrectly accepting the hypothesis H_0 when H_1 is true is

$$\beta = \int_0^{\chi^2} f_1(r^2; n, \lambda)\, dr^2 , \tag{5}$$

where $n = 2p$ and where the upper limit χ^2 is defined by expression (4). The *power of the test*, $\pi = 1 - \beta$, is given by

$$\pi = \int_{\chi^2}^{\infty} f_1(r^2; n, \lambda)\, dr^2 . \tag{6}$$

For track-to-track association the test of H_0 against H_1 is a test of the central chi-square distribution against a noncentral chi-square alternative. Tables relating to this test are available [for example, Ref. 12]. Reference 13 describes a rapid, compact program for carrying out computations based on the noncentral chi-square distribution. This program is suitable for implementation within automated tracking algorithms or for use within computer-driven analyst aids.

Figure 4 indicates the power of the test for levels of significance of 0.10 and 0.01 and for situations involving six and 15 observed positions in each track. When all other parameters are held fixed, the power of the test increases if λ increases, increases if the specified level of significance increases, and decreases if the number of observed positions increases. Generally speaking, the operational situation will determine the values of p, T_0, T_1, T_2, Σ_0, and Σ_1. The error probabilities α and β can be specified. However, the value of λ must be estimated by the surveillance analyst, since the value of the difference vector $\overline{T}_2 - \overline{T}_1$ cannot be assumed to be known. An important question underlying the test procedure is the sensitivity of the resulting errors α and β to variations in the λ value employed. It must be determined whether the test procedure requires great accuracy on the part of the analyst.

Sensitivity to Estimates of λ

The parameter λ is determined by the difference vector $\overline{T}_2 - \overline{T}_1$ and by the matrix sum $\Sigma = \Sigma_0 + \Sigma_1$; since the latter factor will generally be known, the requirement that λ be known reduces to a requirement that the difference vector be estimated, since it cannot be known exactly. The effects of variation in either the difference vector or in λ on the results of the test procedure can be ascertained only by a detailed parametric analysis of the relations between p, α, β, $\overline{T}_2 - \overline{T}_1$, and Σ. Some insights into the general problem of test sensitivity to λ can be obtained from an analysis of the two-candidate test in the following special case. In the two-candidate track association problem the choice of which track to use as a basis for the null hypothesis H_0 is arbitrary. Therefore in the absence of any supplementary information it will be assumed that errors in failing to make a correct association are as important for the alternate hypothesis as for the null hypothesis. In this case it is reasonable to set $\alpha = \beta$. Having specified α, one can proceed as before by finding that value of χ^2 such that equation (4) holds. Using this value as the test criterion, one can then employ equation (5) to determine that value of λ such that the resulting β value is equal to α. Analysis of the resulting relations between $\alpha(=\beta)$ and λ can indicate how precisely λ must be estimated to obtain a test with level of significance α and power $1 - \alpha$.

Figure 5 shows the sensitivity of the resulting test errors to estimates of λ by exhibiting contours along which $\alpha = \beta$ for p values of 6, 15, and 24. Since the power of the test is an increasing function of λ, this figure shows that to guarantee a specified maximum error probability it is necessary only to guarantee that the actual value of λ exceeds a stated threshold. For example, for the case of 15 observed track positions, if one can guarantee that λ is greater than 30, then the test will achieve error rates of less than 8 percent. The lower threshold for estimates of λ increases as the number of observed positions increases and as the allowable maximum error probability decreases. Therefore, although all that the procedure requires is a decision that λ exceed some stated minimum value, it may be difficult to assure that this decision is correct.

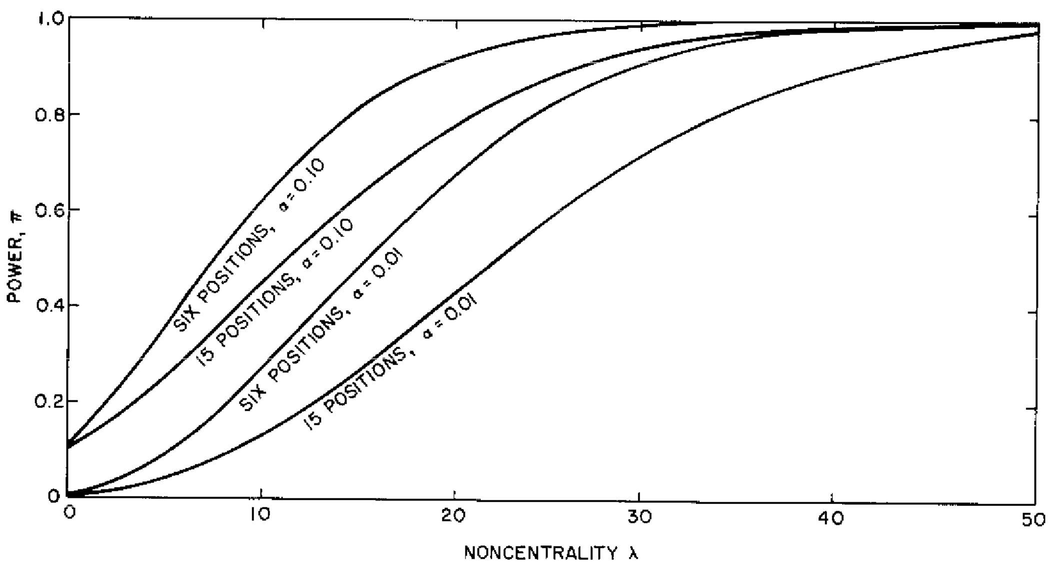

Fig. 4—Power of the test of H_0 against H_1.

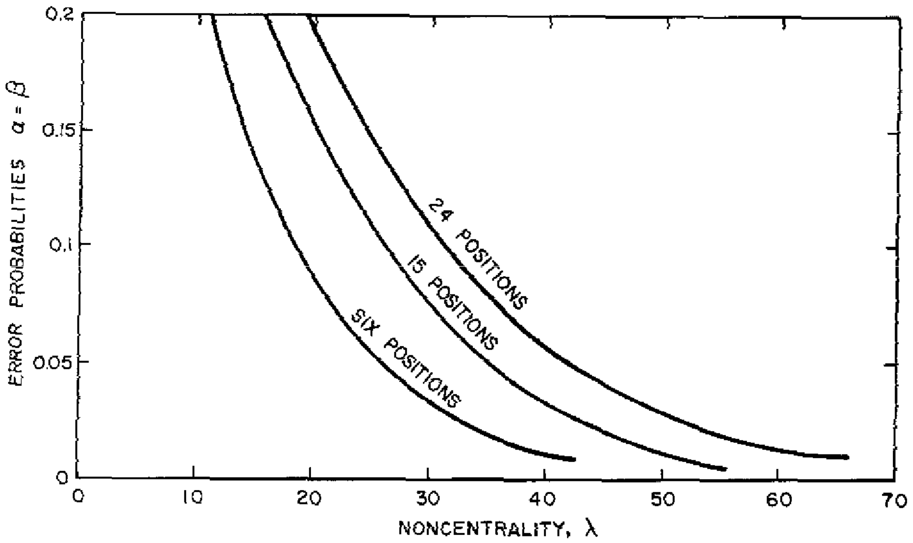

Fig. 5—Sensitivity of the test of H_0 against H_1 to changes in the noncentrality, for $\alpha = \beta$.

An example will illustrate the concepts involved in relating requirements on λ to requirements on the estimated value of $\overline{T}_2 - \overline{T}_1$. Suppose that tracks T_1 and T_2 were observed with a sensor which generated a covariance matrix with no off-diagonal elements and such that both σ_x^2 and σ_y^2 equaled 1 n.mi.^{2} for each observed position. Suppose that T_0 was generated by a system whose covariance matrix also was diagonal, but with $\sigma_x^2 = \sigma_y^2 = 4$ n.mi.^{2} for each position. Finally, suppose six positions (12 coordinates) have been observed and that an analyst can estimate that each element in the difference vector $\overline{T}_2 - \overline{T}_1$ is at least 2.0 n.mi. Then the minimum value for λ will be 9.6, and one can conclude from Fig. 5 that the test could be subject to errors greater than 20 percent. Suppose on the other hand that errors of at most 5 percent were required for tracks comprising 15 observed positions. This would require a value of λ of at least 35, and with the covariance matrix structure described it would be necessary to be correct in deciding that the elements in the difference vector were, on the average, at least 2.4 n.mi.

Computer Implementation

Computer implementation of a procedure based on these results could be readily achieved as an interactive analyst aid. Since the majority of the required parameter values would already be available, an analyst would have to enter his best judgment of the magniture of the elements in the underlying mean difference vector. The program would then generate the test procedure, identifying the operable value of α, computing the value of the upper limit χ^2, computing the value of r^2, and determining whether or not r^2 exceeded χ^2. It would respond to the analyst with a statement of the decision which was made, together with an indication of the error probabilities present in the test used. Conversely, if an analyst were to specify a desired level of significance α, the program could respond with the minimum value of λ required to guarantee an error β no greater than α.

FURTHER ASPECTS OF THE PROBLEM

In this section the preceeding discussion is extended to three additional topics. The first is a method for treating the situation when the basic assumption of simultaneous observations is relaxed. The second is the case in which immediate association decisions are not always required and can be deferred pending the receipt of further data. Finally the third is the importance of selecting proper limits for the error probabilities α and β in the tests and the consequences of making track association errors within a surveillance-data processing system.

Nonsimultaneous Observations

The basic track-to-track association model assumed that the positions constituting the two tracks were observed at the same times. Usually different surveillance systems will make detections at different times; hence for any pair of generated tracks a set of nonsimultaneous observations is more likely than a set of simultaneous observations. The following discussion is a description of a method of testing tracks with nonsimultaneous observations for equality of means. The method transforms the case of nonsimultaneous observations to the case of simultaneous observations. This transformation involves the generation of interpolated points within each track so that both of the resulting new tracks will have contituent points associated with the same set of observation times. The set of observation times contains all times at which either of the two original tracks was observed.

Figures 6 and 7 represent the situation. Figure 6 shows track T_1, observed at times t_2, t_3, t_4, and t_6, and track T_0, observed at times t_1, t_5, and t_7. To test the hypothesis that these tracks relate to the same target, two new tracks T_1^*, and T_0^* have been created (Fig. 7), both of which comprise all the observation times for both T_1 and T_0. The points in the new tracks are of two types. First, all points originally observed for a track are contained in the related new track. Second, if an original track T_k contains two sequential points, observed at times t_i and t_j, and if the other track contains a point observed at an intermediate time t, where $t_i < t < t_j$, then an intermediate point is interpolated between the original points. Letting $D = (t - t_i)/(t_j - t_i)$, the new point is assumed to occur at time t and to be located at a fraction D of the way between the originally observed points. If x_i and x_j are the x coordinates of T_k at times t_i and t_j respectively and x^* is the x coordinate of the new interpolated point in track T_k^* at time t, then

$$x^* = x_i + D(x_j - x_i)$$

$$= \frac{(t_j - t)}{(t_j - t_i)} x_i + \frac{(t - t_i)}{(t_j - t_i)} x_j.$$

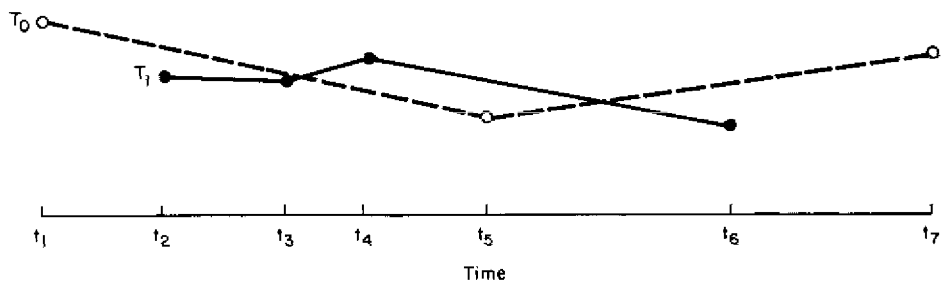

Fig. 6—Tracks with Nonsimultaneous observations

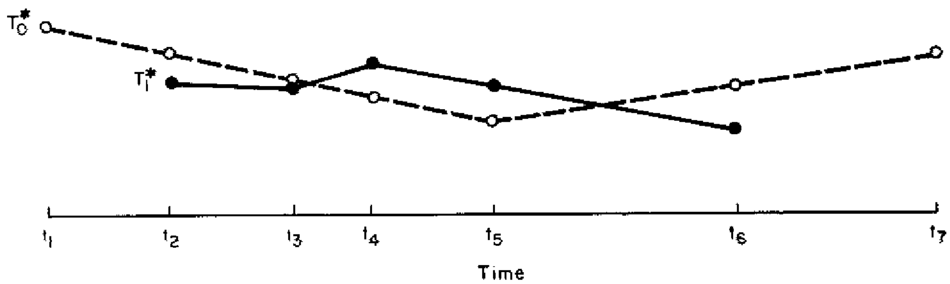

Fig. 7—Nonsimultaneous tracks of Fig. 6 after transformation to the case of simultaneous observations

These three points, the two original ones and the new interpolated one, are all made members of the new track T_k^*. The process is carried out for both of the original tracks and all of the time points for which the interpolation is feasible, resulting in the situation shown in Fig. 7. In this figure the two new tracks T_1^* and T_0^* both have "observation points" at all times t_2 through t_6. Since these points have been obtained by a linear transformation of normal random variables, their related covariance structures can be obtained [8] and they may be used as data for the test of the null hypothesis H_0 described for the basic model. The one-point extensions in track T_0^*, both at the start and at the end, are not used in the hypothesis-testing computations, as the interpolation scheme does not generate extensions from track T_1.

This extension of the basic model to the case of nonsimultaneous observations has been programmed for use within an automated track correlation process. This program is described in Appendix A and listed in Appendix B.

Sequential Tests

A situation was discussed in which track T_0 was to associate with one of two candidate tracks T_1 and T_2 as a result of a testing procedure. It is not always the case that an immediate track association decision is required; quite often the problems inherent in making association errors are severe enough to require that association decisions not be made without sufficient supporting evidence. One method of accomplishing this is through the use of sequential tests based on reexamination of the data set each time new data elements are obtained. In the following discussion the structure of sequential testing procedures within the context of the track association problem is described and some problems inherent in implementing the procedure are indicated.

As described in Refs. 14 and 15, sequential testing procedures are useful in deciding whether observed data correspond to a null hypothesis H_0, whether they correspond to a specified alternate hypothesis H_1, or whether additional data should be obtained before making a final decision. Let $g_{0,n}$ and $g_{1,n}$ be the probability density functions of the multivariate data set $(x_1, x_2, ..., x_n)$ at the nth step of the process, under hypothesis H_0 and H_1 respectively, let

$$h_n = \frac{g_{1,n}(x_1, x_2, ..., x_n)}{g_{0,n}(x_1, x_2, ..., x_n)},$$

and suppose that the error probabilities α and β have been specified. The sequential testing procedure is then defined by the following criteria:

$$\text{if } h_n \leqslant \frac{\beta}{1-\alpha}, \qquad \text{accept } H_0; \tag{7a}$$

$$\text{if } \frac{\beta}{1-\alpha} < h_n < \frac{1-\beta}{\alpha}, \qquad \text{obtain another observation} \tag{7b}$$

$$\text{if } \frac{1-\beta}{\alpha} \leqslant h_n, \qquad \text{accept } H_1. \tag{7c}$$

Note that the criteria do not depend on the sample size n. These criteria were based on the assumptions that the successive observations were stochastically independent samples and that the sequential procedure will, with probability 1, eventually terminate. For a rich family of situations the assumption of stochastically independent samples can be relaxed and the procedure will still be valid. It has been assumed here that the conditions underlying the track association problem are such that the sequential test procedure is valid.

For track association the assumptions of underlying normal distributions imply that under the null hypothesis H_0 specified earlier $T_0 - T_1$ has the density function

$$\begin{aligned} g_{0,n} &= g_0(T_0 - T_1; p) \\ &= \frac{1}{(2\pi)^p |\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(T_0 - T_1)' \Sigma^{-1}(T_0 - T_1)\right] \end{aligned}$$

and under H_1 specified earlier $T_0 - T_1$ has the density function

$$g_{1,n} = g_1(T_0 - T_1; p, \overline{T}_2 - \overline{T}_1)$$

$$= \frac{1}{(2\pi)^p |\Sigma|^{1/2}} \exp\left(-\frac{1}{2} \{[(T_0 - T_1) - (\overline{T}_2 - \overline{T}_1)]' \Sigma^{-1} [(T_0 - T_1) - (\overline{T}_2 - \overline{T}_1)]\}\right).$$

It follows that

$$h_n = \frac{g_{1,n}}{g_{0,n}}$$

$$= \exp\left\{ -\frac{1}{2} [-(T_0 - T_1)' \Sigma^{-1} (\overline{T}_2 - \overline{T}_1) - (\overline{T}_2 - \overline{T}_1)' \Sigma^{-1} (T_0 - T_1) + (\overline{T}_2 - \overline{T}_1)' \Sigma^{-1} (\overline{T}_2 - \overline{T}_1)] \right\}$$

$$= \exp\left\{ -\frac{1}{2} [\lambda - 2(T_0 - T_1)' \Sigma^{-1} (\overline{T}_2 - \overline{T}_1)] \right\}.$$

As discussed previously, it is reasonable to consider that the error probabilities α and β are equal. Thus, with the notation

$$Z = \lambda - 2(T_0 - T_1)' \Sigma^{-1} (\overline{T}_2 - \overline{T}_1),$$

criteria (7) reduce to the following:

$$\text{if } \ell n \left(\frac{1-\alpha}{\alpha}\right)^2 \leqslant Z, \qquad \text{accept } H_0, \tag{8a}$$

$$\text{if } \ell n \left(\frac{\alpha}{1-\alpha}\right)^2 < Z < \ell n \left(\frac{1-\alpha}{\alpha}\right)^2, \qquad \text{wait for additional data,} \tag{8b}$$

$$\text{if } Z \leqslant \ell n \left(\frac{\alpha}{1-\alpha}\right)^2, \qquad \text{accept } H_1. \tag{8c}$$

The performance of this procedure is determined by the relations between the observed values $T_0 - T_1$ and Σ and the estimated mean difference vector $\overline{T}_2 - \overline{T}_1$. Detailed investigations based on realistic predictions of the observed parameter values will be required to ascertain the sensitivity of the sequential procedure to estimates of the difference vector in an operational setting. At this point however examination of the criteria (8) permit general conclusions to be made regarding the test's performance. Figure 8 illustrates these criteria. If the permissible error probabilities are low, the hypothesis acceptance regions are reduced; the sequential procedure will most frequently decide to wait for additional data unless a value of Z is obtained which is large in absolute value. If the test is more lenient—reflected in higher allowable error probabilities—the sequential procedure will more frequently result in a hypothesis acceptance, unless the absolute value of Z is small.

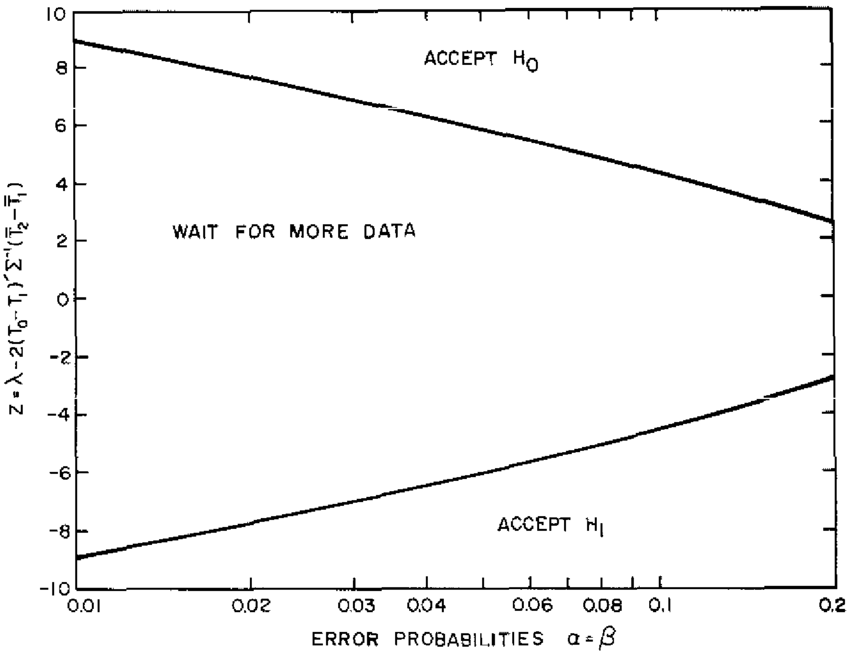

Fig. 8—Sequential test criteria (8) for track association.

In summary, the use of sequential testing procedures may be valuable when two tracks T_1 and T_2 are candidates for association. This test permits additional information to be collected if the available information is adequate for a hypothesis acceptance decision. The test procedure requires the estimation of the mean difference vector $\overline{T}_2 - \overline{T}_1$. The sensitivity of the test's performance to estimates of this value is a function of the observed data $T_0 - T_1$ and covariance matrix Σ; further detailed investigations of this factor will be required to determine the worth and feasibility of implementing the sequential procedure within an operational surveillance processing system.

Specifying the Allowable Error Probabilities

The tests described in this report require the specification of maximum allowable probabilities α and β of incorrect decisions. These errors have real-world costs. Those responsible for implementing these tests in an operational surveillance-data processing system must analyze the consequences of making incorrect decisions in order to specify α and β.

In the case of the basic model the only decision is whether or not to associate a given pair of tracks; as developed, the test requires the specification of α, the probability of incorrectly rejecting the proposed association of the two tracks. Errors of this type will require the surveillance processing system to maintain multiple tracks in situations when maintenance of single tracks would suffice. The data base will thus become cluttered with redundant tracks, and any computational routines that are performed on each track in the data base will be performed more than necessary. More important is the possibility of losing information on special-interest targets which correct track association would have provided. For example, without supporting information on target identities, position-only tracks must all be subject to the same degree of attention by analysts. However, if position-only tracks can be associated with tracks carrying identification information, then known combatant tracks can be specified for special attention and tracks from noncombatants can be specified for lesser attention. Thus a low false rejection rate for H_0 will increase data-processing efficiency and increase the value of the information in the stored tracks. Still the consequences of falsely associating unrelated tracks can be severe. Whereas false rejection results in the loss of useful information, false acceptance surely results in the generation of misleading information. Misclassification of a combatant as a merchant, or conversely, can certainly create false pictures of tactical situations. Track files containing incorrectly associated tracks can easily serve as bases for future compounded errors, thus leading to a possibly useless data base. Extensive lists of the consequences of making either type of error can be generated indefinitely, but these examples should suffice to indicate the importance of proper assessment of the effects of the errors on the operation and ultimate value of the data processing system.

SUMMARY REMARKS

This report has discussed a series of statistical tests for attacking the track-to-track association problem in ocean surveillance. The tests are straightforward and can be readily implemented within many automated surveillance-data processing systems. Computational algorithms exist which permit rapid computation of any of the test statistics involved; consequently these test procedures can be implemented "on line," within an automatic processing system, or they can be used within interactive programs to provide quick response to analyst queries. The demand for inputs from analysts will be at a low level; optimally, analysts may be asked simply to judge whether a given parameter is or is not within a specified range. A major value of these tests lies in their permitting greater use of the various types of information contained in different tracks of the same target. The result of successful implementation of the tests is a data base which is low in errors and low in redundancy.

ACKNOWLEDGMENTS

The author expresses his appreciation to David J. Kaplan for providing numerous helpful suggestions on the report's exposition and to Dr. Joseph H. Kullback for suggesting the problem and providing insights into practical aspects of surveillance-data processing systems.

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Appendix A

A TRACK-TO-TRACK ASSOCIATION PROGRAM

INTRODUCTION

A program for track-to-track association tests is described in this appendix and listed in Appendix B. The program is designed to operate on tracks which are defined by sets of positions and which are accompanied by the underlying covariance matrices. The track positions may comprise locations reported by a surveillance sensor or may consist of estimated positions obtained by applying a smoothing scheme to a set of observations. In the former case the covariance matrix would probably be block diagonal, with two-by-two matrices along the main diagonal; in the latter case the covariance matrix would be a general nonsingular symmetric matrix, possibly with relatively large off-diagonal entries. In any event the multivariate covariance structure underlying the positions constituting the tracks must be known and is to be entered as data. The scheme is based on the assumption that the degree of overhang for overlapping tracks is limited to one point at each end of the tracks; that is, when two tracks start at different times, the early track contains at most one early point and similarly a track which continues after the other has stopped contains at most one late point. Finally the scheme will test track associations for track positions observed either at identical times or at different times.

PROGRAM STRUCTURE

The following steps comprise the major activities of the track association program.

Read Inputs. For each point in each track the program first obtains the time of observation, the X coordinate, and the Y coordinate. In addition, it obtains the covariance matrix related to the set of points constituting each track. Finally it generates a column vector TRK for each observed track, defined such that for the Kth observed track ($K = 1, 2$) the Ith observed (X, Y) coordinate pair is defined as the pair

$$(TRK\,(K, 2I - 1),\ \ TRK\,(K, 2I)).$$

The times T used in the program are assumed to be defined in terms of decimal hours; some changes will be required whenever actual inputs are given in terms of hours, minutes, and seconds.

Compute the Interpolation Parameters. The fundamental concept of the program is the coordinate-by-coordinate comparison of a pair of standardized interpolated tracks. The program obtains these from the track vectors $TRK(1)$ and $TRK(2)$ by a linear interpolation scheme which produces a set of identical time points and related positions within each track. The interpolation scheme proceeds as described in the main body of this report in conjunction with Fig. 6 and 7. Next, if the starting time for one interpolated track is earlier than that for the other one, the program deletes the earlier one, so that both interpolated tracks will now start at the same time; similarly any overhang at the termination of the tracks is reduced to the case of equal track termination time. Thus both interpolated tracks will be based on the same set of observed time points. The number of such points is denoted NPTS; since each observation point is related to two coordinates, the majority of the subsequent computation is based on twice the value of this parameter.

Compute the Transformation Matrices for the Original Tracks. The interpolated tracks used for the statistical test are obtained from the original tracks by linear transformations. If the input track is based on M observed points ($2M$ coordinates), the transformation matrix Q has 2 ($NPTS$) rows and $2M$ columns.

Compute the Interpolated Tracks. For each $K = 1, 2$, input track $TRK(K)$ is transformed into interpolated track $TINT(K)$ via the transformation matrix Q, with suitable dimensions. Since the input track is treated as a column vector, the equation for this transformation is

$$TINT = Q \cdot TRK\,.$$

Compute the Covariance Matrices for the Interpolated Tracks. The covariance matrix Σ_{TI} underlying the related interpolated track is given by

$$\sum_{TI} = Q \sum_{T} Q'\,,$$

where Σ_T is the covariance matrix for an originally obtained track and the prime represents matrix transpose.

Compute the Difference Vector for Interpolated Tracks. The hypothesis that the two original tracks are from the same target is equivalent to the hypothesis that the underlying mean vectors are equal, or that the difference of the mean vectors contains all zero coordinates. This formulation carries over to the interpolated vectors as well, and at this step the program computes the value of the difference $TDIFF$ of the computed interpolated tracks, $TDIFF$ being a column vector.

Compute the Matrix Sum. In addition to the values of the difference vector, the test requires knowledge of the covariance matrix associated with the difference, in this case the sum Σ of the covariance matrices Σ_{TI} associated with the interpolated tracks.

Compute the Measure R^2. R^2 is calculated by the formula

$$R^2 = (TDIFF)'\Sigma^{-1}(TDIFF),$$

where again the prime indicates matrix transpose.

Compute Allowable Upper Limit for R^2. The program next computes the upper limit χ^2, above which computed values of R^2 will lead to rejection of the hypothesis that the two tracks are from the same target. This limit is the solution of expression

(4) of the main body of the report. These computations use two approximations The first one, based on equation (28) in Chapter 17 of Ref. 10 (Vol. 1), permits approximation of the chi-square cumulative distribution function (c.d.f.) by the distribution function of an appropriately defined normal distribution. The second one, based on expression 26.2.23 of Ref. 16, uses the ratio of two polynomials to estimate percentiles of normal distributions. This method requires relatively little execution time and produces results which are within a few percentage points of values in the standard statistical tables.

Make Decision. The program compares the computed value of R^2 with the upper limit χ^2. If $R^2 \leqslant \chi^2$, the program sends a message to merge the two tracks. Otherwise, it sends a message not to merge the tracks based on the available data.

PROGRAM LISTING

Appendix B is a listing of the program, as written for implementation on the CDC KRONOS time-sharing computer system. In the KRONOS version the array sizes reflect the small number of data points used for the test runs; for actual data the array sizes may have to be increased. Also, the following items are specific for KRONOS and will have to be changed for other modes of operation, such as batch processing or implementation within an automated correlation processor:

- The array W1 and W2;
- The CALL MATIDN instruction (line 00160);
- The instruction W2(I, J) = SIGM(I, J) (line 03320);
- The CALL MATINV instruction (line 03660).

ADDITIONAL PROGRAMS

An extended version of this program, which uses Kalman filtering techniques to obtain smoothed position estimates and the related covariance matrix, has been developed. Information on the extended program and on other programs related to the track association problem can be obtained from the author.

Appendix B

PROGRAM LISTING

```
LIST

 75/03/20. 11.28.24.
PROGRAM   ASSOC

PROGRAM ASSOC(OUTPUT,TAPE1)
DIMENSION DUM(16,16), M(2), NUM(2), Q(2,16,16), SG(2,8,8)
DIMENSION SIG(2,16,16), SIGI(16,16), SIGM(16,16), T(8)
DIMENSION TCOM(8), TDIFF(16), TINT(2,16), TM(2,8,8)
DIMENSION TRK(2,8), TS(2,8), X(2,4), Y(2,4)
DIMENSION W1(16,32), W2(16,16)
ALPHA=0.05
CALL MATIDN(W2,16,16)
*
*    INPUT DATA
READ(1,5050) M(1)
5050 FORMAT(I3)
M2=M(1)
M22=M2+M2
DO 20 I=1,M2
READ(1,5070)TS(1,I),X(1,I),Y(1,I)
5070 FORMAT(6F8.4)
20 CONTINUE
DO 30 I=1,M22
READ(1,5070)(SG(1,I,J),J=I,M22)
30 CONTINUE
DO 50 I=2,M22
I1=I-1
DO 40 J=1,I1
SG(1,I,J)=SG(1,J,I)
40 CONTINUE
50 CONTINUE
READ(1,5050) M(2)
M2=M(2)
M22=M2+M2
DO 60 I=1,M2
READ(1,5070) TS(2,I), X(2,I), Y(2,I)
60 CONTINUE
DO 70 I=1,M22
READ(1,5070)(SG(2,I,J), J=I,M22)
70 CONTINUE
DO 90 I=2,M22
I1=I-1
DO 80 J=1,I1
SG(2,I,J)=SG(2,J,I)
80 CONTINUE
90 CONTINUE
DO 100 K=1,2
M2=M(K)
DO 95 I=1,M2
TRK(K,I+I-1)=X(K,I)
TRK(K,I+I)=Y(K,I)
95 CONTINUE
100 CONTINUE
```

```
PRINT 9000
9000 FORMAT(30X,8H1.INPUTS)
DO 140 K=1,2
M2=M(K)
M22=M2+M2
PRINT 9010, K
9010 FORMAT(//,9HTRACK NO.,I2/3X,4HTIME,7X,7HX-COORD,
+5X,7HY-COORD)
DO 130 I=1,M2
PRINT 9020, TS(K,I), X(K,I), Y(K,I)
9020 FORMAT(5(F10.4, 2X))
130 CONTINUE
PRINT 9030,K
9030 FORMAT(//,20X,21HCOVARIANCE MATRIX NO., I2)
DO 135 I=1,M22
PRINT 9040, (SG(K,I,J), J=1,M22)
9040 FORMAT(6(F10.4,1X))
135 CONTINUE
140 CONTINUE
*
*   COMPUTE THE INTERPOLATION PARAMETERS
CALL ORDER(TS,M,T,N)
DO 160 K=1,2
M1=M(K)-1
NUM(K)=0
DO 155 I=1,N
DO 150 J=1,M1
IF ((T(I).GE.TS(K,J)).AND.(T(I).LE.TS(K,J+1))) GO TO 145
GO TO 150
145 TM(K,I,J)=(TS(K,J+1)-T(I))/(TS(K,J+1)-TS(K,J))
TM(K,I,J+1)=(T(I)-TS(K,J))/(TS(K,J+1)-TS(K,J))
NUM(K)=NUM(K)+1
GO TO 155
150 CONTINUE
155 CONTINUE
160 CONTINUE
IF(TS(1,1).EQ.TS(2,1)) GO TO 320
IF(T(1).EQ.TS(1,1)) GO TO 200
I=2
GO TO 240
200 I=1
GO TO 240
240 K=N-1
NUM(I)=NUM(I)-1
DO 300 J=1,K
DO 280 I=1,2
MX=M(I)
DO 260 L=1,MX
TM(I,J,L)=TM(I,J+1,L)
260 CONTINUE
280 CONTINUE
TCOM(J)=T(J+1)
300 CONTINUE
NPTS=MIN0(NUM(1),NUM(2))
GO TO 400
320 NPTS=MIN0(NUM(1),NUM(2))
DO 360 J=1,NPTS
TCOM(J)=T(J)
360 CONTINUE
400 CONTINUE
```

```
*
*    CONSTRUCTION  OF THE MATRICES Q1,Q2 FOR TRANSFORMING
*    TRACKS 1 AND 2,RESPECTIVELY.
DO 740 K=1,2
M2=M(K)
DO 720 I=1,NPTS
DO 700 J=1,M2
INEW=I+I-1
JNEW=J+J-1
Q(K,INEW,JNEW)=TM(K,I,J)
INEW=I+I
JNEW=J+J
Q(K,INEW,JNEW)=TM(K,I,J)
700 CONTINUE
720 CONTINUE
740 CONTINUE
*
*    COMPUTING THE INTERPOLATED TRACK POSITIONS.
NROW=NPTS+NPTS
DO 800 K=1,2
NCOL=M(K)+M(K)
DO 780 I=1,NROW
TINT(K,I)=0.
DO 760 J=1,NCOL
TINT(K,I)=TINT(K,I)+(Q(K,I,J)*TRK(K,J))
760 CONTINUE
780 CONTINUE
800 CONTINUE
*
*    COMPUTATION OF SIG=Q*SG*(Q-TRANSPOSE)
DO 900 K=1,2
NCOL=M(K)+M(K)
DO 880 I=1,NROW
DO 860 J=1,NROW
SIG(K,I,J)=0.
DO 840 N=1,NCOL
DUM(I,N)=0.
DO 820 L=1,NCOL
DUM(I,N)=DUM(I,N)+(Q(K,I,L)*SG(K,L,N))
820 CONTINUE
SIG(K,I,J)=SIG(K,I,J)+(DUM(I,N)*Q(K,J,N))
840 CONTINUE
860 CONTINUE
880 CONTINUE
900 CONTINUE
NCOL=NROW
```

```
*
*    COMPUTE TDIFF=TINT1-TINT2, DIFFERENCE VECTOR
DO 920 I=1,NROW
TDIFF(I)=TINT(1,I)-TINT(2,I)
920 CONTINUE
*
*    COMPUTE SIGM=SIG1+SIG2, MATRIX SUM
DO 960 I=1,NROW
DO 940 J=1,NCOL
SIGM(I,J)=SIG(1,I,J)+SIG(2,I,J)
W2(I,J)=SIGM(I,J)
940 CONTINUE
960 CONTINUE
PRINT 9050
9050 FORMAT(//,20X,20H2.TRANSFORMED VALUES/19X,7HTRACK 1,
+17X,7HTRACK 2/3X,4HTIME,2X,2(5X,7HX-COORD,5X,7HY-COORD))
DO 965 I=1,NPTS
PRINT 9020,TCOM(I),TINT(1,2*I-1),TINT(1,2*I),TINT(2,2*I-1),
+TINT(2,2*I)
965 CONTINUE
PRINT 9060
9060 FORMAT(//,15X,38HSUM OF TRANSFORMED COVARIANCE MATRICES)
DO 970 I=1,NROW
PRINT 9040,(SIGM(I,J), J=1,NROW)
970 CONTINUE
*
*    COMPUTE R2=(TDIFF-TRANSPOSE)*(SIGM-INVERSE)*(TDIFF)
CALL MATINV(W2,SIGI,16,32,W1)
R2=0.
DO 1000 J=1,NROW
DUM(1,J)=0.
DO 980 I=1,NROW
DUM(1,J)=DUM(1,J)+(TDIFF(I)*SIGI(I,J))
980 CONTINUE
R2=R2+(DUM(1,J)*TDIFF(J))
1000 CONTINUE
PRINT 8000, R2
8000 FORMAT(//,20X,17H3.COMPUTED VALUES,/3HR2=,F12.5)
```

```
*
*    COMPUTE THE (1-ALPHA)-TH PERCENTILE POINT FOR THE
*    CHI-SQUARE DISTRIBUTION WITH 2*NPTS DEGREES OF FREEDOM
XNPT=2*NPTS
PROB=1.-ALPHA
IF(PROB.LE.0.5) GO TO 1020
P=1.-PROB
GO TO 1040
1020 P=PROB
1040 TE=SQRT(ALOG(1./(P*P)))
C0=2.515517
C1=0.802853
C2=0.010328
D1=1.432788
D2=0.189269
D3=0.001308
U1=1.+(D1*TE)+(D2*TE*TE)+(D3*TE*TE*TE)
U2=C0+(C1*TE)+(C2*TE*TE)
XP=TE-(U2/U1)
XP=XP*(PROB-0.5)/(ABS(PROB-0.5))
A=(XP*SQRT(2./(9.*XNPT)))+1.
CHISQ=XNPT*((A-(2./(9.*XNPT)))**3)
PRINT 8020,ALPHA,XNPT,CHISQ
8020 FORMAT(6HALPHA=, F6.3, 2X, 15HDEG. OF FREEDOM,F4.0,
+2X,6HCHISQ=, F12.4)
*
*    DECISION POINT
IF(R2-CHISQ) 1060,1060,1080
1060 PRINT 8040
8040 FORMAT(12HMERGE TRACKS)
GO TO 2000
1080 PRINT 8060
8060 FORMAT(32HDO NOT MERGE TRACKS AT THIS TIME)
GO TO 2000
2000 CONTINUE
STOP
END
```

```
SUBROUTINE ORDER(Q,MM,H,N)
DIMENSION Q(2,15),MM(2),H(15),XX(15)
MMI=MM(1)
DO 20 I=1,MMI
XX(I)=Q(1,I)
20 CONTINUE
MMM=MM(1)+MM(2)
MM2=MM(2)
DO 40 I=1,MM2
J=MM(1)+I
XX(J)=Q(2,I)
40 CONTINUE
DO 80 J=1,MMM
H(J)=XX(J)
DO 60 I=J,MMM
IF (XX(I).GE.H(J)) GO TO 60
Z=H(J)
H(J)=XX(I)
XX(I)=Z
60 CONTINUE
80 CONTINUE
N=MMM
DO 120 I=1,MMM
IF (H(I).NE.H(I+1)) GO TO 120
I1=I+1
DO 100 J=I1,MMM
H(J)=H(J+1)
100 CONTINUE
N=N-1
MMM=MMM-1
120 CONTINUE
MMM=MM(1)+MM(2)
RETURN
END
```